Simple and Robust Risk Budgeting with Expected Shortfall

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The twin problems of measuring and allocating risk have a long and storied history going back to the seminal work of Markowitz [1956, 1959]. The following six decades have seen widespread acceptance of Markowitz's mean–variance optimization paradigm, though, in recent years, some users have complained that this paradigm fails to live up to its promise when put into practice. The majority of complaints about the paradigm are centered around its robustness to errors in its inputs and to its failure to account for tail risk, while the majority of proposed solutions address one or more of its deficiencies with some sacrifice of its simplicity and elegance; see, for example, Jobson and Korkie [1981], Black and Litterman [1992], Ceria and Stubbs [2006], and Michaud and Michaud [2008].

In spite of the complaints directed at it, the mean–variance paradigm has captured the hearts and minds of investment professionals for a good reason; it captures the essential aspects of the investor's problem in a parsimonious way, particularly when optimizing risk-controlled portfolios without derivatives.

In this article, we describe a simple enhancement to the mean–variance paradigm that preserves its simplicity and its intuitive appeal, while accounting for risk in a more sophisticated way. In particular, we usefully extend the closed-form mean–variance risk budgeting solutions of Blitz and Hottinga [2001] and Berkelaar, Kobor, and Tsumagiri [2006] to account for tail risk. Our methodology charts a pragmatic middle course between the mathematical elegance of mean–variance optimization and the technical difficulties of tail risk optimization.

The remainder of this article is organized as follows. We first provide a brief introduction to coherent measures of risk and introduce three coherent measures that are of particular interest to us: expected shortfall, multiple-environment average expected shortfall, and multiple-environment maximum expected shortfall. We then review Blitz and Hottinga's [2001] solution to the problem of allocating tracking error among independent alpha sources and show how it can be enhanced by substituting a coherent measure of risk for variance. Following this, we similarly extend and enhance Berkelaar, Kobor, and Tsumagiri's [2006] solution to the problem of budgeting tracking error among correlated alpha sources, and then illustrate the benefits of our approach with a numerical example that is drawn from actual practice. Finally, we summarize our results, and point to various avenues for future research.

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COHERENT MEASURES OF RISK AND EXPECTED SHORTFALL

Markowitz's [1959] association of risk with the variance of security returns was pragmatic; that is, variance is readily estimated from historical returns, and the variance of a portfolio is related to the variance of its constituents via a quadratic form. But while variance is a good measure of risk in a world of normally distributed returns, it is not a good measure for the risk of assets with significant tail risk as is the case with many fixed-income securities and with options. Various measures of risk, some ad hoc, others less so, have therefore gained currency with practitioners. Value at risk, or VaR, is perhaps the best known such measure (Jorion [2006]) and is defined as follows:

$$VAR_{\alpha}(X) = -\sup[x \mid P[X \leq x] \leq (1-\alpha)] \qquad (1)$$

In words, $VAR_{\alpha}(X)$ is the greatest lower bound on the loss one can experience with probability no greater than $1 - \alpha$. While many criticisms of VaR, such as those of Taleb [1997] and Taleb and Jorion [2007], relate to the fact that it must be estimated from historical data and cannot therefore account for the risk of events that have not already occurred, a very different criticism of it emerged in a seminal paper by Artzner, Delbaen, Eber and Heath [1977].

In a fundamental break with Markowitz, Artzner et al. [1997] defined risk to be a measure of the set of possible future values of an asset relative to a riskless asset, and not of the uncertainty in its return. If $\rho(X)$ is a risk measure for a portfolio X, positive values of $\rho(X)$ can be thought of as the minimum amount of riskless capital that must be added to X in order to support it, while negative values of $\rho(X)$ can be thought of as the maximum amount of capital that can be removed from the portfolio without making it unacceptable to a regulator or risk manager.

Unfortunately, the mathematical formalism required to fully develop this view obscures its powerful economic intuition. We, therefore, refer the mathematically inclined reader to Artzner et al. [1997] and focus here instead on an intuitive explanation of their ideas. Artzner et al. asserted that a good risk measure should possess the following properties:

1. Monotonicity: If $X < Y$ almost surely, then $\rho(X) \geq \rho(Y)$
2. Translation Invariance: For any constant c, $\rho(X + c) = \rho(X) - c$
3. Positive Homogeneity: For any positive constant c, $\rho(c \times X) = c \times \rho(X)$
4. Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$

The first property is related to the theory of stochastic dominance. If one asset is almost surely dominated by another asset, the risk of the first asset must necessarily be higher. The second property asserts that if a riskless asset of value c is added to (subtracted from) a position, its risk must decrease (increase) by the same amount. The third property asserts that risk scales linearly with position size, and the last property asserts that risk cannot be created by combining portfolios.

The first three properties seem reasonable enough, but the fourth requires explanation. It is rooted in the observation that under some measures of risk, it is sometimes possible to partition a risky portfolio into two or more subportfolios, each of which appears to have little risk, or even no risk at all, as the following example from Artzner et al. [1997] shows. Consider two portfolios, one of which is short an out-of-the-money European put, and the other which is short an out-of-the-money European call, both on the S&P 500, and both maturing on the same day. Assume that both the call and the put will be exercised with a probability of 4%. The 95% VaR of both portfolios at maturity is therefore zero. Now, combine the two portfolios. The probability that at least one of the two options will be exercised is 8%, and the 95% VaR is therefore greater than zero. Combining the portfolios appears to have created risk instead of diversifying it away!

While this example might seem contrived, the problem of noncoherence is not. We regularly see risk reports from well-respected commercial vendors in which the marginal VaR of an asset or a strategy exceeds its VaR. In spite of this, VaR is not easily discarded or dismissed because it has a certain intuitive appeal to portfolio managers, traders, and risk managers, and is deeply ingrained in many risk management processes, particularly those associated with trading desks. What then, is the smallest change that we can make to VaR to ensure its coherence while preserving (or better still, enhancing) its focus on tail risk?

Delbaen [2000] and Acerbi and Tasche [2002] showed that for a large class of continuous distributions without atoms, which satisfy a certain technical

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condition (the Fatou property), expected shortfall at probability level α, defined as

$$ES_\alpha(X) = -E[X \mid X < -VaR_\alpha(X)] \tag{2}$$

is the smallest coherent majorant to VaR, provided the expectation exists. This definition of expected shortfall can be extended to all distributions for which the expectation exists, but the mathematical machinery required in the general case obscures its powerful economic intuition. We refer the interested reader to Delbaen [2000] and Acerbi and Tasche [2002] for details. Expected shortfall can also be defined in terms of the cumulative distribution of returns $F_X(\cdot)$ via

$$ES_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha F_X^{-1}(x)dx \tag{3}$$

Though many coherent risk measures have been proposed in the literature, it is expected shortfall that has received the most attention for the following reasons:

1. It captures our intuitive notion that risks lurk in the tail of a distribution and is readily interpreted as an estimate of the average loss that we will experience with a given probability.
2. It is easily estimated via simulation.
3. It the smallest coherent majorant to VaR.

In practice, it proves impossible to evaluate the integral in Equation (3), and we therefore estimate expected shortfall by averaging the most negative outcomes of a simulation. Let $X_1, X_2, \ldots, X_N$ be N independent realizations of X, and let $X_{1:N}, X_{2:N}, \ldots, X_{N:N}$ be their order statistics (i.e., their values arranged in ascending order). We estimate expected shortfall at probability level α via

$$\hat{ES}_\alpha(X) = -\frac{\sum_{1 \le i < (1-\alpha)N} X_{i:N}}{(1-\alpha)N} \tag{4}$$

With our definitions and our intuition in place, we can now state the following proposition, a simple proof of which is provided in the appendix.

Proposition 1: *Expected shortfall is a coherent measure of risk (Acerbi and Tasche [2002]).*

Most risk measures, expected shortfall included, are computed under a single distribution $F_X(\cdot)$ that is estimated from the current investment environment, with current prices, and at current levels of volatility and correlation. But, as time passes, and as the investment environment changes, asset prices, volatilities, and correlations will change and so, correspondingly, will the risk of a portfolio. This uncertainty in the investment environment can be addressed by computing a coherent measure of risk under a family of distributions, each member of which is associated with a particular investment environment that we might encounter in the future, and then combining these results to derive a new coherent risk measure that provides us a better understanding of the risk of a portfolio. Computing a risk measure under a family of distributions protects against a particular form of model risk, because the distribution of returns is part of the model. We next describe two such risk measures that are based on expected shortfall, and which we call multiple-environment average expected shortfall (or *MEAES*) and multiple-environment maximum expected shortfall (or *MEMES*).

Formally, we assume that in the future the investment environment s will be drawn from a countable set of investment environments S with probability p_s. We further assume that in environment s, the conditional distribution of asset returns will be $F_{X|S}(\cdot)$. Let $ES_\alpha(X|s)$ be the conditional Expected Shortfall at probability level α in this environment. We define $MEAES_\alpha(X)$ and $MEMES_\alpha(X)$ to be the expectation and the maximum, respectively, of the conditional expected shortfall over the set of all investment environments we might encounter, that is,

$$MEAES_\alpha(X) = E_S\{ES_\alpha(X \mid s)\} \tag{5}$$

and

$$MEMES_\alpha(X) = \max_{s \in S}\{ES_\alpha(X \mid s)\} \tag{6}$$

The following proposition, whose proof is deferred to the appendix, addresses the coherence of *MEAES* and *MEMES*.

Proposition 2: *If all the investment environments have positive measure, MEAES and MEMES are coherent measures of risk.*

In practice, we cannot enumerate all possible investment environments, and so we instead restrict S to a relatively small set of plausible environments that we assume are equally probable: one in which interest rates rise or fall a fair bit, another in which yield curves steepen or flatten a fair bit, a third in which credit spreads widen or narrow a fair bit, a fourth in which volatility increases or decreases a fair bit, a fifth in which equities move a fair bit, and so on. These choices are not unique, and different users will make different choices, both for the environments and for their probabilities, depending on the composition of their portfolios, and on the risks they are exposed to. The 10 most recent calendar years are an equally viable set of environments, and may be preferred by some.

We interpret "a fair bit" to mean "lying in the top or bottom 10% of all daily moves that have occurred over the past decade." This choice is arbitrary, but pragmatic. We want the environments we choose to be representative of environments that we are likely to encounter, and thus we deliberately avoid focusing on extreme scenarios such as market crashes that, in our view, are the province of stress tests and not of risk measures.

We estimate $ES_\alpha(X|s)$ as follows. We first identify historical periods associated with investment environment s—for example, the set of all days on which yield curves steepened or flattened by at least 10 basis points (bps)—then use data from this subperiod to estimate the conditional joint distribution of assets, and finally compute the conditional expected shortfall via a simulation. We repeat this procedure for each of our chosen environments, then average our results to obtain an estimate of $MEAES_\alpha(X)$ and take the maximum to obtain an estimate of $MEAES_\alpha(X)$, that is,

$$M\hat{E}AES_\alpha(X) \cong -\frac{1}{|S|}\sum_{s\in S}\left[\frac{\sum_{1\le i<(1-\alpha)N} X_{i:N} \mid s}{(1-\alpha)N}\right] \tag{7}$$

and

$$M\hat{E}MES_\alpha(X) \cong \max_{s\in S} -\frac{\sum_{1\le i<(1-\alpha)N} X_{i:N} \mid s}{(1-\alpha)N} \tag{8}$$

We set $\alpha = 0.99$ and $N = 3000$, and so derive these estimates from approximately 30 values in each state. We find empirically that with six equi-probable states of the investment environment, and for fixed-income portfolios without significant tail risk, $M\hat{E}AES_\alpha(X)$ and $M\hat{E}MES_\alpha(X)$ are approximately 50% and 100% larger, respectively, than $\hat{E}S_\alpha(X)$. For portfolios with significant tail risk, however, this ratio increases, giving us advance warning of risks that are not immediately evident, and also alerting us to the particular market conditions under which the portfolio could suffer significant losses, thus providing us with a simple, though useful, form of reverse stress testing.

Exhibit 1 illustrates the risk profile, as well as the $M\hat{E}AES_{0.99}$ and $M\hat{E}MES_{0.99}$, of a global fixed-income portfolio that is run by seven portfolio managers under six assumed states of the investment environment. For reasons of confidentiality, we suppress information on the actual holdings, but display the types of trades in the portfolio. The following scenarios are considered:

1. CDS indices move more than 5 bps in one day
2. Major currencies (EUR or JPY) move over 1.5% in one day
3. The six months following the collapse of Lehman Brothers
4. The yield on the on-the-run 10-yr. U.S. Treasury moves more than 10 bps in one day
5. The slope of the U.S. or EU yield curve moves more than 5 bps in one day
6. U.S. swap spreads move than 4 bps in one day

A number of observations can be made from this exhibit.

1. For the entire portfolio, $M\hat{E}AES_{0.99}$ and $M\hat{E}MES_{0.99}$ are about 25% and 50% larger, respectively, than $\hat{E}S_{0.99}$. Averaged across all portfolio managers, they are about 83% and 122% larger, respectively, than $\hat{E}S_{0.99}$, clearly pointing to beneficial correlations between the various strategies in a wide range of environments. This suggests that *MEAES* and *MEMES* can be used directly to allocate risk capital to strategies, portfolios, and trading desks without applying an ad hoc scaling factor to VaR as suggested in the Revised Basel II Market Risk Framework.
2. The portfolios of Portfolio Manager 2 and Portfolio Manager 7 have hidden tail risks; that is, their expected shortfall is higher in every single environment than it is in the current environment. Furthermore, the increase in risk for both portfolios is substantially larger than it is for all the other portfolios. This suggests that their holdings ought to be reviewed and the proximate causes of their sharp rise in risk in other investment environments be identified.

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EXHIBIT 1
Risk Profile of a Global Fixed-Income Portfolio

	ES (99% 1 day)	MEMES	MEAES	CDX or iTraxx over 5 bps 1d move	JPY or EUR over 1.5% 1d move	Post Lehman Period (6 mths to Mar 2009)	U.S. 10-yr over 10 bps 1d move	U.S. or EUR 2s10s over 5 bps 1d move	U.S. SP10 over 4 bps 1d move	Worst Scenario
Entire Portfolio	$46,867	$71,728	$59,064	$52,689	$64,099	$71,728	$54,241	$47,060	$64,569	Post Lehman Period (6 mths to Mar 2009)
Portfolio Manager 1	$8	$18	$14	$14	$13	$18	$12	$11	$16	Post Lehman Period (6 mths to Mar 2009)
PM 1 Country Spread	$8	$18	$14	$14	$13	$18	$12	$11	$16	Post Lehman Period (6 mths to Mar 2009)
Portfolio Manager 2	$9,088	$34,469	$28,328	$27,375	$28,741	$25,776	$34,469	$26,515	$27,094	U.S. 10-yr over 10 bps 1d move
PM 2 Duration	$456	$961	$670	$608	$961	$713	$578	$555	$606	JPY or EUR over 1.5% 1d move
PM 2 Inflation	$7,578	$27,807	$22,293	$20,141	$20,590	$21,741	$27,807	$19,271	$24,209	U.S. 10-yr over 10 bps 1d move
PM 2 Yield Curve	$8,232	$21,172	$18,701	$21,172	$17,827	$20,642	$15,177	$18,032	$19,358	CDX or iTraxx over 5 bps 1d move
Cash Management	$2,259	$4,853	$3,409	$3,130	$4,853	$3,642	$2,926	$2,807	$3,097	JPY or EUR over 1.5% 1d move
Portfolio Manager 3	$7,926	$11,666	$10,596	$11,666	$11,270	$10,560	$10,993	$9,622	$9,465	CDX or iTraxx over 5 bps 1d move
PM 3 Country Spread	$3,216	$7,140	$6,089	$7,026	$5,831	$6,015	$5,489	$7,140	$5,032	U.S. or EUR 2s10s over 5 bps 1d move
PM 3 Duration	$6,948	$8,921	$7,773	$7,841	$8,921	$7,476	$8,573	$5,984	$7,844	JPY or EUR over 1.5% 1d move
Portfolio Manager 4	$26,181	$27,636	$19,917	$18,813	$19,504	$27,636	$15,510	$16,565	$21,476	Post Lehman Period (6 mths to Mar 2009)
PM 4 Country Spread	$26,181	$27,636	$19,917	$18,813	$19,504	$27,636	$15,510	$16,565	$21,476	Post Lehman Period (6 mths to Mar 2009)
Portfolio Manager 5	$21,150	$44,607	$38,043	$32,673	$42,472	$37,107	$44,607	$31,030	$40,365	U.S. 10-yr over 10 bps 1d move
PM 5 Duration	$23,996	$41,431	$33,952	$28,813	$35,574	$33,086	$41,431	$29,955	$34,851	U.S. 10-yr over 10 bps 1d move
PM 5 Inflation	$7,635	$19,205	$16,315	$14,148	$17,946	$19,205	$16,045	$13,333	$17,216	Post Lehman Period (6 mths to Mar 2009)
PM 5 Volatility	$33	$47	$40	$44	$39	$47	$39	$33	$41	Post Lehman Period (6 mths to Mar 2009)
Portfolio Manager 6	$18,104	$34,597	$27,605	$26,353	$27,982	$34,597	$28,378	$22,914	$25,407	Post Lehman Period (6 mths to Mar 2009)
PM 6 Duration	$388	$670	$592	$621	$606	$670	$489	$510	$654	Post Lehman Period (6 mths to Mar 2009)
PM 6 Inflation	$18,098	$34,711	$27,695	$26,481	$28,082	$34,711	$28,515	$23,060	$25,320	Post Lehman Period (6 mths to Mar 2009)
Portfolio Manager 7	$12,607	$38,120	$30,946	$26,435	$31,892	$38,120	$28,958	$26,065	$34,205	Post Lehman Period (6 mths to Mar 2009)
PM 7 Inflation	$7,614	$19,280	$16,359	$14,218	$17,916	$19,280	$16,117	$13,471	$17,152	Post Lehman Period (6 mths to Mar 2009)
PM 7 Swap Spreads	$3,306	$10,292	$8,118	$8,302	$7,261	$10,292	$6,152	$6,909	$9,790	Post Lehman Period (6 mths to Mar 2009)
PM 7 Volatility	$757	$1,467	$1,218	$1,412	$1,280	$1,467	$1,011	$978	$1,430	Post Lehman Period (6 mths to Mar 2009)
PM 7 Yield Curve	$12,062	$24,461	$20,340	$17,613	$19,408	$24,461	$19,690	$18,385	$22,481	Post Lehman Period (6 mths to Mar 2009)

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3. Portfolio Manager 4 is running a portfolio whose risk is very high in the current environment, and which will decrease in most other environments considered. This is unusual, and once again, is worth an exploratory investigation.
4. The portfolio managers are well diversified and are, in fact, close to independent of each other in spite of the fact that they take risk in similar markets; the expected shortfall of the portfolio ($46,867) is less than one-third the sum of the individual expected shortfalls ($166,460) and is well approximated by the square root of the sum of their squares ($42,071).
5. Not all portfolios achieve their maximum risk in the post-Lehman period. Portfolio Managers 2, 3, and 5 will see their risk maximized in various other states of the world.

APPLICATION TO RISK BUDGETING

Even though coherent risk measures have made enormous inroads into our thinking about risk, they have had little impact on the allocation of risk because the (coherent) risk of a portfolio is not related in a simple way to that of its constituent assets, precluding the formulation of a convex optimization problem. There have been two broad classes of attempts to resolve this issue.

1. Some authors, such as Rachev et al. [2009], estimated parametric forms for the distributions of asset returns and from these obtained a parametric form for the distribution of a portfolio's return, which they then optimized via a convex optimization.
2. Others, such as Rockafellar and Uryasev [2000] and Chekhlov, Uryasev, and Zabarankin [2005], used simulation techniques to sample asset returns and showed that the portfolio optimization problem can then be structured as a linear program. In particular, Rockafellar and Uryasev simultaneously computed the VaR of a portfolio and minimized its conditional VaR subject to a set of linear constraints.

While both approaches promise enhanced allocations of risk, neither has gained broad currency among asset managers, although the first approach has made inroads into the management of funds of hedge funds. We attribute this in large part to the fact that these approaches are sufficiently complex that they require either a significant programming effort by a mathematically sophisticated user or the purchase of a commercial implementation of the algorithm.

This is characteristic of most robust risk allocation algorithms. It proves difficult, if not impossible, to solve them in closed form, even in the simplest case when there are only two assets, making it correspondingly difficult to build an intuitive understanding of their inner working and of the properties of an optimal portfolio.

Our approach, in contrast, is exceptionally simple to understand and implement, and in many cases of practical interest can be solved on the back of an envelope. Instead of solving a complex optimization problem, we first create a mean-variance optimal risk budget (i.e., we use tracking error as our measure of risk), and then modify it in a simple way that preserves its tracking error, but which results in a reduction in its tail risk when asset returns are not normally distributed. When asset returns *are* normally distributed, our modification simply recovers the mean-variance optimal risk budget that we started with.

It is, to the best of our knowledge, the first robust risk budgeting method that admits a simple closed-form solution, and it inherits its solvability from our pragmatic decision to keep one foot in the old world (we start by formulating a risk budget using variance as our measure of risk) and the other in the new (at an appropriate point, we switch to a coherent measure of risk). Its utility lies in the fact that it modifies the mean-variance solution in an intuitive way that preserves its optimality in some important special cases, and which strikes portfolio managers and risk managers as being both logical and desirable in others.

The starting point for our method is a particular form of the classical mean-variance solution obtained by Blitz and Hottinga [2001]. Given a set of N independent alpha strategies with information ratios IR_i (defined to be the ratio of the expected excess return to the standard deviation of excess returns, also known as tracking error, $IR_i \equiv \frac{\mu_i}{\sigma_i}$), and a target tracking error of σ_{Target}, Blitz and Hottinga showed that the optimal allocation of tracking error to the ith strategy is

$$\sigma_i^{Alloc} = \frac{IR_i}{\sqrt{\sum_{1 \le j \le N} IR_j^2}} \times \sigma_{\text{Target}} \qquad (9)$$

Tracking error is allocated to a strategy in proportion to its information ratio, which is naturally

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interpreted in a mean-variance world as the price of risk. If all information ratios are positive, all risk allocations are correspondingly positive. If we do not know the prospective information ratios of all available strategies, but have views on their relative levels (i.e., Strategy 2's information ratio is twice that of Strategy 1), Equation (9) still holds because scaling each information ratio by the same constant does not change the solution. In the special case when all strategies have the same information ratio, risk is allocated equally among them.

To model the impact of tail risk, we define the modified information ratio of a strategy to be

$$\begin{aligned} IR_i^* &= \frac{\mu_i}{ES_{\alpha i}} \\ &= IR_i \times \frac{\sigma_i}{ES_{\alpha i}} \end{aligned} \tag{10}$$

We use expected shortfall in Equation (10) to clarify and make concrete the development of our ideas. We could just as well have used *MEAES*, *MEMES*, or any other coherent measure of risk, but have deliberately chosen not to in the interest of simplicity. If the expected shortfall is a scalar multiple of volatility (as it is for normally distributed random variables), the modified information ratio is also a scalar multiple of the information ratio. We now make a simple ad hoc modification to Equation (9). We replace IR_i with IR_i^*, and write

$$\begin{aligned} \sigma_i^{Alloc} &= \frac{IR_i^*}{\sqrt{\sum_{1\le j\le N} IR_j^{*2}}} \times \sigma_{\text{Target}} \\ &= \frac{IR_i \dfrac{\sigma_i}{ES_{\alpha i}}}{\sqrt{\sum_{1\le j\le N} IR_j^{\,2} \times \left(\dfrac{\sigma_j}{ES_{\alpha j}}\right)^2}} \times \sigma_{\text{Target}} \end{aligned} \tag{11}$$

On comparing Equations (9) and (11), we see that they both result in the same total tracking error, but with (potentially) different allocations of risk. If the expected shortfall is a constant scalar multiple of the standard deviation, as it is for normally distributed random variables, our solution is identical to Blitz and Hottinga's [2001] solution, and therefore inherits its mean-variance optimality. If it is not, our solution will tilt the risk allocation away from strategies with high levels of tail risk and toward strategies with low levels of tail risk, which is exactly what a thoughtful portfolio manager would do in practice. Given our focus on tail risk, one might ask why we do not directly allocate expected shortfall among strategies; that is, why do we not allocate expected shortfall according to the following equation?

$$ES_{\alpha i}^{Alloc} = \frac{IR_i^*}{\sqrt{\sum_{1\le j\le N} IR_j^{*2}}} \times ES_{\text{Target}} \tag{12}$$

While seemingly attractive, Equation (12) suffers from two problems:

1. There is no guarantee that the target level of expected shortfall will be achieved, because the expected shortfall of a portfolio is not related to that of its constituent assets in an analytically tractable way.
2. Most investors specify their investment objectives in terms of tracking error, which is easily estimated from monthly performance data, rather than in terms of expected shortfall, which is not.

If it is imperative that expected shortfall, and not tracking error, be allocated among competing alpha strategies, it will prove necessary to compute a mean expected-shortfall efficient portfolio using one of the methods mentioned earlier. It is our experience, though, that any theoretical gains from a more precise mathematical formulation are quickly dissipated in the quicksand of parameter estimation, and it is for precisely this reason that most robust risk allocation algorithms have found little use in portfolio management.

Our approach has something in common with the Black–Litterman [1992] method in that both approaches tilt a mean-variance efficient allocation in accordance with some auxiliary information. The key difference between them lies in the fact that Black and Litterman tilted (and in fact, optimized) their initial allocation in accordance with auxiliary information about expected returns, while we tilt, but do not optimize, our initial allocation in accordance with auxiliary information about risk.

A particularly useful form of Equation (11) results when all the strategies are assumed to have the same information ratio. In this case, risk is allocated according to

$$\sigma_i^{Alloc} = \frac{\frac{\sigma_i}{ES_{\alpha i}}}{\sqrt{\sum_{1 \le j \le N} \left(\frac{\sigma_i}{ES_{\alpha i}} \right)^2}} \times \sigma_{\text{Target}} \tag{13}$$

We use Equations (11) and (13) extensively when allocating tracking error to independent alpha strategies for global multi-sector fixed-income portfolios. Their recommended allocations make intuitive sense to portfolio managers and risk managers alike, and have proven stable over time. The most common question we are asked is how we would treat correlated alpha sources. Unfortunately, there is no simple way in which to account for dependencies when using a coherent measure of risk, but once again, a few modifications to an existing mean-variance solution prove useful.

Berkelaar, Kobor, and Tsumagiri [2006] expanded on Blitz and Hottinga's results and showed that tracking error is optimally allocated to correlated alpha sources according to

$$\sigma^{Alloc} = \frac{\rho^{-1} IR}{\sqrt{IR' \; \rho^{-1} \; IR}} \times \sigma_{\text{Target}} \tag{14}$$

where σ^{Alloc} is a column vector of tracking error allocations, IR is a column vector of information ratios, IR' is its transpose, and ρ is the correlation matrix of the strategies. The denominator of Equation (14) can be shown to be the information ratio of the optimal portfolio. In addition, setting ρ to the identity matrix in Equation (14) recovers Equation (9), as it should. To adapt Equation (14) to a coherent measure of risk, we again replace information ratios by modified information ratios and write

$$\sigma_i^{Alloc} = \frac{\rho^{-1} IR^*}{\sqrt{IR^{*'} \; \rho^{-1} IR^*}} \times \sigma_{\text{Target}} \tag{15}$$

where IR^* is a column vector of modified information ratios, and $IR^{*\prime}$ is its transpose. Equations (14) and (15) are analogous to Equations (9) and (11), and likewise result in the same tracking error, but with potentially different allocations of risk. For all distributions for which the expected shortfall is a scalar multiple of the standard deviation, we recover the mean-variance efficient risk allocation, and for distributions that do not satisfy this condition, we tilt the mean-variance efficient risk allocation in a way that strikes portfolio managers and risk managers as being both logical and desirable.

Equations (14) and (15) do not guarantee the positivity of our risk allocations. While this can be resolved by solving a constrained quadratic program, we then lose the elegant simplicity of our solution. We propose instead to shrink the correlation matrix by replacing each off-diagonal element by the arithmetic average of all its off-diagonal elements, and to then solve Equation (15) with the shrunk correlation matrix. The structure of the inverse makes it very unlikely that any of the risk allocations will turn negative.

Elton and Gruber [1973] and Elton, Gruber, and Urich [1978] were early advocates of this model and showed that it provided better estimates of future correlations than the sample correlation matrix. Ledoit and Wolf [2004] explored the optimal level of shrinkage when shrinking the sample covariance matrix to the constant average correlation matrix, and our proposal may be thought of as an extreme implementation of their proposal. Henderson and Searle [1981] showed that the inverse of the constant average correlation matrix is

$$\begin{bmatrix} 1 & \bar{\rho} & . & \bar{\rho} \\ \bar{\rho} & . & . & . \\ . & . & . & \bar{\rho} \\ \bar{\rho} & . & \bar{\rho} & 1 \end{bmatrix}^{-1} = \frac{1}{(1-\bar{\rho})(1+(N-1)\bar{\rho})} \times \begin{bmatrix} 1+(N-2)\bar{\rho} & -\bar{\rho} & . & -\bar{\rho} \\ -\bar{\rho} & . & . & . \\ . & . & . & -\bar{\rho} \\ -\bar{\rho} & . & -\bar{\rho} & 1+(N-2)\bar{\rho} \end{bmatrix} \tag{16}$$

where $\bar{\rho}$ is the arithmetic average of all the off-diagonal elements in the actual correlation matrix ρ.

The simple structure of the inverse makes it possible to solve to Equation (15) in a spreadsheet, or even by hand, with only a little more effort than in the independent case. Gratifyingly, this simple modification consistently results in intuitively reasonable and strictly positive solutions. This is not surprising because the diagonal elements of the inverse are much larger than its off-diagonal elements, and it therefore takes a high average correlation or a very imbalanced set of modified information ratios to make one or more of the

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allocations negative. Neither situation occurs frequently in practice; in fact, we find that our alpha sources are nearly independent. Consequently, Equations (11) and (13) satisfy our needs much of the time, and when an assumption of independence seems too strict, we turn to Equations (15) and (16). We have not felt compelled to solve a constrained quadratic program.

The average correlation can be obtained from the simulations used to estimate tail risk or estimated from a time series of strategy returns. The following theorem is the basis of an exceptionally simple and robust estimator of the average correlation from a time series of returns. It requires one only to count the number of strategies with positive and negative returns in each time period, and it can be used to estimate the average correlation over any period of interest. Its proof is deferred to the appendix.

Proposition 3: *Consider a set of N jointly normally distributed random variables with a mean of zero, and for which we have available their contemporaneous returns. Let the number of assets with positive returns in time period t be denoted* N_t^+*, and the number of assets with negative returns in time period t be denoted* N_t^-*. Then,* $\hat{\rho}_t \equiv 1 - 4 \times \frac{N_t^+ \times N_t^-}{N(N-1)}$ *is a point estimate of the instantaneous average cross-sectional correlation between these N random variables in time period t.*

This estimator of average cross-sectional correlation is robust to outliers, because it does not depend on the magnitude of individual security returns, but only on their signs. It can be further averaged over time to obtain the average correlation over any interval of interest, or exponentially weighted to get a more dynamic estimate of average correlation. The assumption of normality greatly simplifies the proof of Proposition 3 and is not a liability in practice—simulations show the estimator to work well for a wide range of distributions.

AN EXAMPLE

We have successfully applied this methodology to fixed-income portfolio management, and the following example illustrates our use of it. An active fixed-income portfolio is decomposed into a beta replication component and four independent alpha strategies: Interest Rates, Structured Securities (MBS and CMBS), Sector Rotation, and Foreign Exchange. The alpha strategies, which are created by independent alpha teams, are carefully hedged so that they have as little market risk as possible, and are only weakly correlated with each other, as can be seen from the current correlation matrix in Exhibit 2; the average of all the off-diagonal elements is 0.017. These correlations are estimated using RiskMetrics RiskManager™, a commercial risk management software package.

EXHIBIT 2
Forecasted Correlations between Four Alpha Strategies

	Sector Rotation	Structured Securities	Interest Rates	Foreign Exchange
Sector Rotation	1.00	–0.07	0.32	–0.15
Structured Securities	–0.07	1.00	0.10	–0.21
Interest Rates	0.32	0.10	1.00	0.11
Foreign Exchange	–0.15	–0.21	0.11	1.00

We assume that all four strategies have the same information ratio. We have no ex ante evidence to rank order these alpha teams. We, therefore, allocate tracking error among them in accordance with Equations (15) and (16), and with $IR_i = 1$, $1 \le i \le 4$. We use expected shortfall as our coherent measure of risk only in order to facilitate the replication of our results. If we had used *MEAES* or *MEMES* in its place, the relative allocations would shift in accordance with the tail risk of each strategy, but the tracking error of the portfolio would remain constant.

Tracking error and expected shortfall, like correlations, are computed using RiskMetrics RiskManager™, and we call the resulting measure TEES (an acronym for Tracking Error over Expected Shortfall). Our results are displayed in Exhibit 3, and it takes only a cursory glance to identify those strategies that will be underweighted and those that will be overweighted.

The mean-variance solution allocates a substantial amount of risk to the Foreign Exchange (FX) strategy in spite of its high level of tail risk, because it is negatively correlated with the Structured Securities and Sector Rotation strategies. The TEES methodology, in contrast, takes into account the tail risk of the FX strategy, and substantially lowers its risk budget without any need to artificially constrain it. The sum of all four allocations is 1.94, which is slightly lower than 2, the level that one would expect if all four strategies were independent and jointly normally distributed; our risk budgeting method does not induce unwanted leverage. The sum of all four allocations in the mean-variance solution, in contrast, is 2.08.

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Exhibit 3

Risk Allocation among Four Alpha Strategies Using TEES

	Sector Rotation	Structured Securities	Interest Rates	Foreign Exchange
Tracking Error (1 day, bps)	1.00	1.82	0.95	1.78
Expected Shortfall (99% 1 day, bps)	2.40	4.55	3.61	7.10
TEES (Tracking Error/Expected Shortfall)	0.42	0.40	0.26	0.25
Mean-Variance Allocation of Tracking Error	58%	66%	15%	69%
Allocation of Tracking Error Using TEES	61%	59%	38%	36%

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Exhibit 4

Comparison of Risk Measures under Two Risk Budgeting Methodologies

	Mean-Variance Risk Budget	Risk Budget Using TEES
Tracking Error (1 day, bps)	2.5	2.4
Expected Shortfall (99% 1 day, bps)	8.4	7.1
TEES (Tracking Error/Expected Shortfall)	0.30	0.34

The tracking error and the expected shortfall of the total portfolio computed using both traditional mean-variance risk budgeting and our robust risk budgeting approach are displayed in Exhibit 4. The tracking error depends only weakly on the allocation method. In a perfect world, it would be independent of the allocation method, but minor differences always emerge in simulations, particularly when the various strategies hold assets with nonlinear payoffs. The expected shortfall, however, is about 15% lower when allocating risk using Equations (15) and (16), and the ratio of tracking error to expected shortfall is correspondingly about 13% higher, clearly illustrating the benefits of our approach. In effect, our procedure naturally guides a portfolio manager in the same direction that a conscientious risk manager would. Furthermore, all our computations can be replicated on the back of the proverbial envelope or in a spreadsheet without any need for proprietary software.

SUMMARY

We have shown how coherent measures of risk such as expected shortfall, multiple-environment average expected shortfall, and multiple-environment maximum expected shortfall can be used to enhance a mean-variance risk budget in a simple and intuitive way. This is, to the best of our knowledge, the first robust risk budgeting method that can be solved in closed form on the back of an envelope. It inherits its closed-form solvability from a pragmatic compromise we make: we keep one foot in the "old world" (we start by using variance as our measure of risk) and plant the other firmly in the "new world" (at an appropriate point, we switch to using a coherent measure of risk). It works well in practice in spite of its simplicity, and it yields fixed-income risk allocations that reflect both a portfolio manager's and risk manager's intuition better than does a standard mean-variance risk budget. It is natural to ask if there is a class of (non-multivariate normal) distributions for which it is provably optimal, and we leave this to future research.

Appendix

Proof of Proposition 1

Expected shortfall is a coherent measure of risk.

We verify each of the four properties of a coherent measure of risk.

1. If $X < Y$ almost surely, then $X_{i:N} < Y_{i:N}$ almost surely. It follows that $ES_\alpha(X) \geq ES_\alpha(Y)$
2. $ES_\alpha(X+c) = -\lim_{N\to\infty} \dfrac{\sum_{1\leq i<(1-\alpha)N} X_{i:N} + c}{(1-\alpha)N} = ES_\alpha(X) - c$
3. $ES_\alpha(X\times c) = -\lim_{N\to\infty} \dfrac{\sum_{1\leq i<(1-\alpha)N} X_{i:N} \times c}{(1-\alpha)N} = c \times ES_\alpha(X)$

4. $$ES_\alpha(X+Y) = -\lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}(X+Y)_{i:N}}{(1-\alpha)N}$$
$$\le -\lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}X_{i:N}}{(1-\alpha)N} - \lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}Y_{i:N}}{(1-\alpha)N}$$
$$= ES_\alpha(X) + ES_\alpha$$

Q.E.D.

Proof of Proposition 2

Identical to proof of Proposition 1. We verify each of the four properties of a coherent measure of risk, first, for *MEAES*, and then for *MEMES*.

MEAES

1. Because all the investment environments chosen are defined on sets of positive measure, $X < Y$ a.s., $\Leftrightarrow X|s \le Y|s$ a.s., it follows that $MEAES_\alpha(X) \ge MEAES_\alpha(Y)$.

2. $$MEAES_\alpha(X+c) = -\sum_s p_s \lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}X_{i:N}\mid s+c}{(1-\alpha)N}$$
$$= MEAES_\alpha(X) - c$$

3. $$MEAES_\alpha(X\times c) = -\sum_s p_s \lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}X_{i:N}\mid s\times c}{(1-\alpha)N}$$
$$= c\times MEAES_\alpha(X)$$

4. $$MEAES_\alpha(X+Y) = -\sum_s p_s \lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}(X+Y)_{i:N}\mid s}{(1-\alpha)N}$$
$$\le -\sum_s p_s \lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}X_{i:N}\mid s}{(1-\alpha)N}$$
$$-\sum_s p_s \lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}Y_{i:N}\mid s}{(1-\alpha)N}$$
$$= MEAES_\alpha(X) + MEAES_\alpha(Y)$$

MEMES

1. Once again, because all the investment environments chosen are defined on sets of positive measure, $X < Y$ a.s., $\Leftrightarrow X|s \le Y|s$ a.s., it follows that $MEMES_\alpha(X) \ge MEMES_\alpha(Y)$.

2. $$MEMES_\alpha(X+c) = \max_{s\in S} -\lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}X_{i:N}\mid s+c}{(1-\alpha)N}$$
$$= MEMES_\alpha(X) - c$$

3. $$MEMES_\alpha(X\times c) = \max_{s\in S} -\lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}c\times X_{i:N}\mid s}{(1-\alpha)N}$$
$$= c\times MEMES_\alpha(X)$$

4. In each environment s, and for each i, $-(X+Y)_{i::N}|s \le -[X_{i::N}|s + Y_{i::N}|s]$
Therefore,

$$MEMES_\alpha(X+Y) = \max_{s\in S} -\lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}(X+Y)_{i:N}|s}{(1-\alpha)N}$$
$$\le \max_{s\in S} -\lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}X_{i:N}|s}{(1-\alpha)N}$$
$$+\max_{s\in S} -\lim_{N\to\infty}\frac{\sum_{1\le i<(1-\alpha)N}Y_{i:N}|s}{(1-\alpha)N}$$
$$= MEMES_\alpha(X) + MEMES_\alpha(Y)$$

Q.E.D.

Proof of Proposition 3

Assume we have time series of returns for N jointly normal random variables, as follows:

$$\begin{aligned}\{r_1\} &= r_{10}, r_{11}, \ldots, r_{1t}, \ldots\\ \vdots &= \vdots\\ \{r_N\} &= r_{N0}, r_{N1}, \ldots, r_{Nt}, \ldots\end{aligned} \quad \text{(A-1)}$$

The periodicity of the time series is not material, but in practice we would set it to the shortest interval for which all the returns are synchronous (i.e., for which the returns overlap each other). For convenience, we assume that these time series have a time-invariant mean of zero, but this is not crucial to what follows; if their mean is non-zero, we can simply take first differences to force each mean to zero. Consider any two time series, say $\{r_i\}$ and $\{r_j\}$, and observe them both at time t. It can be shown that the probability that they have opposite signs is given by

$$P[r_{it}\times r_{jt} < 0] = \frac{\cos^{-1}\rho_{ij}}{\pi} \quad \text{(A-2)}$$

where ρ_{ij} is the correlation between random variables i and j. This can be rewritten as

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$$\rho_{ij} = \cos(\pi \times E[1 - u(r_{it} \times r_{jt})]) \quad \text{(A-3)}$$

where $u(x)$ is the unit step function (i.e., $u(x) = 1$ if $x > 0$, and is 0 otherwise), and $E[\cdot]$ denotes expectation. At each instant of time t, and for each pair of securities i and j, we can write

$$\hat{\rho}_{ijt} = \cos(\pi \times (1 - u(r_{it} \times r_{jt}))) \quad \text{(A-4)}$$

But $\cos(0) = 1$, and $\cos(\pi) = -1$, so that Equation (A-4) reduces to

$$\hat{\rho}_{ijt} = 2 \times u(r_{it} \times r_{jt}) - 1 \quad \text{(A-5)}$$

(i.e., $\hat{\rho}_{ijt} = 1$ if the two returns have the same sign and -1 if they have opposite signs). For small values of ρ_{ij}, the time average of $\hat{\rho}_{ijt}$ is an approximately unbiased estimate of ρ_{ij} as in this case $E(1 - u(r_{it} \times r_{jt})) \approx 0.5$, and $\cos(\pi x)$ is approximately linear with a slope of 1 around $x = 0.5$.

Finally, let N_t^+ securities have positive returns, and N_t^- securities have negative returns in time period t. Clearly, $N = N_t^+ + N_t^-$. The average of $\hat{\rho}_{ijt}$ over all pairs of securities at time t is therefore

$$\hat{\rho}_t = E_t[\hat{\rho}_{ijt}] = \frac{\frac{N_t^+(N_t^+-1)}{2} + \frac{N_t^-(N_t^--1)}{2} - N_t^+ \times N_t^-}{\frac{N(N-1)}{2}}$$
$$= 1 - 4 \times \frac{N_t^+ \times N_t^-}{N(N-1)} \quad \text{(A-6)}$$

Q.E.D.

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