Portfolio Insurance with Adaptive Protection (PIWAP)

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Abstract:

This article investigates the optimal design and management of portfolio insurance for target date funds. Capital protection is often set at 100% at inception for simplicity's sake, but without any clearer rationale. We propose a framework for estimating the optimal level of protection, or equivalently, the optimal level of the cushion that maximizes investor utility while taking into account the aversion of that same investor to risk or loss. The optimal management rule that we call Portfolio Insurance with Adaptive Protection (PIWAP) offers the right trade-off between upside potential and capital protection at the target date. Under this strategy, the capital protection could increase over time so that the cushion is capped at a pre-defined level.

The appetite for funds with capital protection has been increasing in recent years. This is particularly the case for individuals saving for retirement and can be explained by the growth in defined contribution pension schemes and the rollercoaster performance of equity markets in the last two decades. Upside potential remains important but not at any cost, and the protection of a certain minimum level of capital is demanded. We thus investigate the optimal design and management of portfolio insurance.

Constant Proportion Portfolio Insurance (CPPI) has traditionally been employed in the management of funds offering insurance on the capital invested. These funds invest a portion of the raised capital in fixed income assets to make sure that at least some of this capital will be recovered at a given future target date. The remainder of the capital is invested in the risky asset in order to generate performance. This latter portion of capital is larger when the cushion which represents the maximum loss not jeopardizing the capital protection is larger. The cushion depends on the discounting factor to the target date of the fund and the level of the capital protection expected at the target date. Capital protection has been traditionally set at 100% of the capital invested. It is not difficult to understand the attraction of this, i.e. a guarantee of no losses on the capital invested. However, there is no financial rationale behind this level of protection. The recent low levels of interest rates have exposed the weakness of this choice since almost all capital must now be committed to fixed income if the 100% capital protection level is still to be pursued. Some fund managers had to start offering levels of protection on the capital invested of below 100% in order to have a sufficiently large cushion and upside potential.

In this article we address this issue and propose a framework for estimating the optimal level of protection or the optimal level of the cushion, which are two sides of the same coin. Too small a cushion offers insufficient upside potential. A cushion that is too large can lead to unacceptably large losses and offers a unattractive level of protection. The optimal insurance strategy we propose increases the protection of the invested capital up to the amount that keeps the cushion under a sufficient level, which will be a function of the remaining time and of the aversion of the investor to risk or loss. This insurance strategy, which we call Portfolio Insurance with Adaptive Protection (PIWAP), is easy to implement and offers the right trade-off between upside potential and protection at the target date.

The first section of this paper reviews key insights from financial literature on CPPI strategies. The second section introduces the theoretical framework for the optimality of the portfolio insurance strategy. This section also derives the optimal size of the cushion as a function of the risk aversion of the investor and the initial protection at target date. The third section takes into account the fact that the level of protection at target date should not be set once and for all but should instead be allowed to increase in the future. In the conclusion we highlight the key innovations proposed in this paper for designing strategies with capital protection.

CPPI STRATEGIES

Strategies with capital protection at a target date are shown to be optimal when the utility function includes an expected upside potential and an exogenous minimum level of subsistence. Kingston (1989) shows explicitly the optimality of such strategies by applying the results of Merton (1971). It is of interest to note that the expected upside potential in those papers is modeled using a utility function involving inter-temporal consumption.

Strategies with capital protection at a target date are typically managed using CPPI approaches with a pre-defined bond floor and adjust the exposure to the risky asset α dynamically in such way as to avoid losing more than the cushion C existing between the net asset value NAV and the bond floor. The exposure to the risky asset follows the equation:

$$\alpha = mC \quad (1)$$

where m is called the multiplier. The optimal multiplier m is shown to be proportional (or even equal in the case of a logarithmic utility function - see below for further details) to the ratio of the expected Sharpe ratio SR and the volatility σ:

$$m = SR / \sigma \tag{2}$$

The optimal multiplier m has been derived in continuous time. Sellers of portfolio insurance have to reset the exposure to the risky asset to m times the size of the cushion at discrete points in time due to operational constraints. Let us define the gap size as the ratio of the cushion to the risky asset, i.e. $C / \alpha = 1 / m$. If the risky asset falls by more than the gap size between two rebalancing dates then the loss is larger than the cushion and the value of the portfolio falls below the bond floor, thereby jeopardizing the protection at maturity. The risk of losing more than the cushion between two rebalancing dates and hence failing to ensure the protection at maturity is known as the gap risk. The larger the multiplier, the larger the gap risk.

To the best of our knowledge, no optimal multiplier has yet been derived in discrete time. Black and Perold (1992) extensively investigate the properties of CPPI strategies in discrete time. They show that if an exposure limit to the risky asset is imposed then when the multiplier m approaches infinity the expected return of the CPPI strategies approaches that of a stop-loss strategy. The expected return of the CPPI strategy is shown to be larger for finite m than for infinite m. They also show that keeping the multiplier small is important in order to limit the gap risk. The multiplier must be lower than the inverse of the maximum drawdown in the risky asset between two dates if one is to ensure the protection at the target date. Working in discrete time leads to another issue related to the fact that the risk of the risky asset, e.g. equities, is not constant over time. The maximum drawdown is thus likely to change according to changes in the risk of the risky asset. Ameur and Prigent (2011) and

Hamidi, Maillet and Prigent (2012) discuss the question of whether the multiplier and/or cushion should be set once and for all or should be adjusted dynamically taking into account the changes in the risk of the risky asset. We shall not address those questions since we consider the continuous time case only.

To our knowledge the question of how to set the level of protection at the target date and how to increase this level of protection has never been fully discussed. Boulier and Kanniganti (1995) investigate empirically the impact of a ratchet increasing the level of capital protection at the target date but they do not address the question from a theoretical point of view. Knowing that CPPI strategies have been designed with utility functions, it is actually surprising that the optimal initial capital protection and the optimal ratchet increasing the protection are not yet known. In this article we address these questions from a theoretical point of view in what follows.

UTILITY FUNCTIONS

Strategies offering capital protection monitor the value of the protection discounted with the yield of the zero coupon bond maturing at the target date. The current value of the protection calculated this way is called the bond floor. Investing the bond floor in the relevant zero-coupon bond delivers the protected capital at the target date. The difference between the *NAV* and the bond floor is the cushion which can be invested in the risky asset in order to generate returns in excess of the protected capital. The risky asset is often leveraged.

Let us assume the following utility function for investors in such strategies:

$$U[NAV(T), G(T)] = P[G(T)] + \lambda \cdot OP[NAV(T) - G(T)] \qquad (3)$$

The first term $P[G(T)]$ represents the utility of the guarantee $G(T)$, i.e. the protected capital, at maturity T. The second term $OP[NAV(T) - G(T)]$ represents the utility of the over-performance relative to the guarantee. The constant λ drives the relative importance of the protection and of the over-performance in the utility function. The larger the risk aversion, the smaller the weight λ of the over-performance in the utility function. $1/\lambda$ can thus be seen as a measure of the investor risk aversion or aversion to loss.

Let us assume that $G(T)$ is chosen when the capital is first invested and not changed ever after. It is then more convenient to derive a closed form solution to the problem of maximizing utility, which is easier to analyze. This assumption will be lifted in the last section of this article in order to resolve the more general optimization program. Maximizing the utility function then involves only the second term in equation (3):

$$\mathrm{Max}\, E_t\{OP[NAV(T)-G(T)]\} = \mathrm{Max}\, E_t\{OP[C(T)]\} = J[t, C(t)] \tag{4}$$

with $C(T)$ the cushion of the strategy at the target date, $E_t\{\}$ the expected value at time t and J the value function. Let us assume that at each time t we can invest α_t in a risky asset with a price $S(t)$ that follows the Brownian dynamics:

$$dS(t) = (r+\Pi)S(t)dt + \sigma\, S(t)dW_t \tag{5}$$

where Π is the asset risk premium, σ the risky asset volatility and r the risk free rate. Let us also assume that the interest rate risk related to ensuring the protection is hedged so that the change of the cushion size is simply driven by the following process:

$$dC(t) = \alpha(t)[(r+\Pi)dt + \sigma\, dW_t] + (C(t)-\alpha(t))rdt = [\alpha(t)\Pi + rC(t)]dt + \alpha(t)\sigma dW_t \tag{6}$$

The value function $J[t, C(t)] = E_t\{OP[C(T)]\}$ is the solution of the Hamilton-Jacobi-Bellman equation which can thus be written as follows:

$$dJ = J_t + \underset{\alpha}{\mathrm{Max}}\left\{[\alpha\Pi + rC(t)]J_C + \frac{1}{2}\sigma^2\alpha^2 J_{CC}\right\} = 0 \tag{7}$$

where the subscripts represent the derivatives of the value function J. The optimal allocation to the risky asset is given by the maximum of the second order equation in (7):

$$\alpha^* = -\frac{\Pi}{\sigma^2}\frac{J_C}{J_{CC}} \tag{8}$$

The value function is therefore characterized by the following Hamilton-Jacobi-Bellman equation and the final condition at time T:

$$\begin{cases} J_t - \dfrac{\Pi^2}{2\sigma^2}\dfrac{(J_C)^2}{J_{CC}} + rCJ_C = 0 \\ \\ J(T,C) = OP(C) \\ \end{cases} \tag{9}$$

Logarithmic utility function

Assuming a logarithmic utility function $OP[C(T)] = \ln[C(T)]$ leads to the following value function:

$$J[t,C(t)] = \ln[C(t)] + \left(r + \frac{\Pi^2}{2\sigma^2} \right)(T-t) \tag{10}$$

as a solution of the system of equations (9). The allocation to the risky asset is then obtained by maximizing equation (8):

$$\alpha^*[t,C(t)] = \frac{\Pi}{\sigma^2}C(t) \tag{11}$$

Equation (11) shows that the optimal cushion should be levered by a multiplier $m = \Pi/\sigma^2 = SR/\sigma$ with SR the Sharpe ratio of the risky asset. The exposure to the risky asset is then equal to mC as in equation (1) and (2). Note that the optimal leverage is inversely proportional to the volatility. The optimal risk budget for each unit of cushion, $m\sigma$, is thus equal to the Sharpe ratio of the risky asset.

Constant Relative Risk Aversion (CRRA) utility function

Assuming a CRRA utility function $OP[C(T)] = C(T)^{\gamma}/\gamma$ where $\gamma \in]0;1[$ leads to the following value function:

$$J[t, C(t)] = \frac{C(t)^{\gamma}}{\gamma} \exp\left[\left(r\gamma + \frac{\Pi^2}{2\sigma^2}\frac{\gamma}{1-\gamma}\right)(T-t)\right] = \frac{C^F(t)^{\gamma}}{\gamma} \exp\left[\left(\frac{\Pi^2}{2\sigma^2}\frac{\gamma}{1-\gamma}\right)(T-t)\right] \quad (12)$$

as a solution of the system of equations (9). The expected utility is the product of the forward cushion to the power gamma times the exponential of a linear function of time. The optimal allocation to the risky asset is then obtained using equation (8):

$$\alpha^*[t, C(t)] = \frac{\Pi}{\sigma^2(1-\gamma)} C(t) \quad (13)$$

The optimal strategy is thus similar to the optimal strategy found for the logarithmic utility function in equation (11) but with different leverage. The optimal cushion should now be levered by a multiplier $m = SR/(\sigma(1-\gamma))$. The leverage is now a function of $1-\gamma$ which is a measure of the risk aversion. The largest aversion to risk ($1-\gamma = 1$ or $\gamma = 0$) corresponds to the logarithmic utility case with $m = SR/\sigma$. Increasing γ reduces the risk aversion and increases the leverage as a result. The smallest possible aversion to risk ($1-\gamma = 0$ or $\gamma = 1$) corresponds to the risk neutral case in the limit of an infinite multiple. Note that the aversion to risk is captured by $1-\gamma$ whereas the aversion to loss is captured by $1/\lambda$.

SETTING THE INITIAL PROTECTION

The expected utility J of the cushion has been determined above by finding the optimal allocation to the risky asset from the time $t = 0$ to the time $t = T$. The protection level could be seen as fixed because it was assumed to be set for once and for all at the time $t = 0$. The optimal initial level of protection can now be determined by looking at the optimization problem at time $t = 0$ and by using the expected utility of the cushion found above for the two types of utility function considered. Note that the optimal protection found in this section assumes that the protection is set a time $t = 0$ and never increased later.

Logarithmic utility

In the case of a logarithmic utility function this leads to the maximization of the following total utility at time $t = 0$:

$$\begin{aligned} E_0(U[NAV(T)]) &= \ln G(T) + \lambda \cdot E_0[\ln C(T)] \\ &= \ln G(T) + \lambda \cdot J[0, C(0)] \\ &= \ln G(T) + \lambda \cdot \ln C(0) + \lambda(r + \Pi/2\sigma^2)(T - t) \end{aligned} \tag{14}$$

The last term in (14) can be dropped when looking for the optimal initial cushion at time $t = 0$. The initial protection and cushion are related by the following equation:

$$C(0) = NAV(0) - G(T)e^{-rT} = NAV(0) - G(T)DF(0,T) \tag{15}$$

where $DF(t,T)$ is the discount factor, i.e. the discounted value at time t of 1 (in units of local currency) received at time T. Taking into account this relationship in (15) and taking the derivatives of equation (14) gives the following optimal initial cushion at time $t = 0$:

$$C(0) = \frac{NAV(0)}{1 + 1/\lambda} \tag{16}$$

Note that the optimal initial cushion is constant over time since it is neither a function of the time to target date nor a function of the interest rate. The optimal initial protection set at time t = 0 is thus:

$$G(T) = \frac{NAV(0)}{DF(0,T)} \frac{1}{1+\lambda} = \frac{NAV^F(0)}{1+\lambda} \tag{17}$$

where the forward $NAV^F(0)$ is what an investor gets paid by investing 100% of the initial capital $NAV(0)$ in the zero coupon bond maturing at the target date. The higher the interest rate the larger the forward $NAV(0)$ and the larger the optimal protection. Note that the optimal protection is a function of both interest rates and aversion to loss. The optimal protection is thus not necessarily 100% of the capital invested.

Setting the aversion to loss to its maximum ($1/\lambda = +\infty$ or $\lambda = 0$) leads, as expected, to a protection equal to the forward $NAV^F(0)$ and hence to invest 100% of the initial capital in the zero coupon bond maturing at the target date. No investment is made in the risky asset as the weight of the over-performance in the utility function in equation (3) is set to $\lambda = 0$.

Constant Relative Risk Aversion (CRRA) utility

Considering the CRRA utility function**:**

$$U[NAV(T), G(T)] = \frac{G(T)^\gamma}{\gamma} + \lambda \cdot \frac{C(T)^\gamma}{\gamma} \tag{18}$$

the maximization of the total utility at time $t = 0$ leads to the optimal initial cushion:

$$C(0) = \frac{NAV(0)}{1 + [\lambda \cdot f(0,T)]^{1/(\gamma-1)}} \tag{19}$$

and to the initial protection:

$$G(T)=\frac{NAV(0)}{DF(0,T)}\frac{[\lambda\cdot f(0,T)]^{1/(\gamma-1)}}{1+[\lambda\cdot f(0,T)]^{1/(\gamma-1)}}=NAV^F(0)\frac{[\lambda\cdot f(0,T)]^{1/(\gamma-1)}}{1+[\lambda\cdot f(0,T)]^{1/(\gamma-1)}} \tag{20}$$

where:

$$f(t,T)=\exp\left[(T-t)\frac{\Pi^2}{2\sigma^2}\frac{\gamma}{1-\gamma}\right] \tag{21}$$

The optimal initial level of the cushion is no longer constant as in the case of a logarithmic utility function: it is an increasing function of the time left to the target date T and of the Sharpe ratio of the risky asset Π/σ. It is intuitive: the longer the time left before the target date, the larger the expected performance from investing in the risky asset, the larger the initial cushion and investment in the risky asset as a result.

In figure 1 and figure 2 we show the initial optimal cushion and initial protection as a function of the time to target date. Two levels of interest rate, 1% and 5%, were used and two levels for the Sharpe ratio of the risky asset, 0.15 and 0.40, were considered. The parameters in the CRRA utility functions were set at $\gamma = 0.50$ and $\lambda = 0.25$.

In figure 1 we find that the initial cushion is larger when the time to target date is longer. The initial cushion also increases with the Sharpe ratio. This is due to the fact that the performance of the cushion is expected to be higher if the remaining time is longer or if the Sharpe ratio is larger.

In figure 2 we find a strong impact of interest rates on the initial protection. Higher interest rates lead to larger initial protection. Note that the optimal initial protection is below 100% most of the time when interest rates are low. Figure 2 also shows that the optimal initial protection actually ends up decreasing with the time to target date when the Sharpe ratio of the risky asset is large.

The optimal strategy has been determined assuming that the protection is set at the time $t = 0$ and not changed afterwards. The strategy is therefore dynamic only as far as the allocation to the risky asset is concerned but not as far as the protection is concerned. One may nevertheless remove this constraint and be willing to increase the protection at a later stage above the protection initially set. The question is whether it is better to let the cushion increase to higher levels while keeping the protection at its initial optimal value or to reduce the size of the cushion when it becomes too large in order to increase the protection, i.e. to *click,* in practitioners' jargon.

INCREASING THE PROTECTION IN THE FUTURE

Let us consider two investors with the same target date but allocating the capital to the optimal strategy defined above at different times. Let us also assume that the first investor experiences a good performance and the cushion increases to levels much above its initial optimal level at time $t = 0$ by the time the second investor decides to invest. It is obvious that the second investor does not care about the past. The optimal strategy for the second investor is thus one where the cushion is likely to be lower than the cushion reached by the optimal strategy followed by the first investor up to that point. But both investors have the same target date. Thus, if it is optimal for the second investor to have a lower cushion than that found in the strategy of the first investor at that particular point in time then it is likely that the first investor should consider reducing the cushion to increase the level of protection.

For the logarithmic utility function, the cushion is likely to be capped at a level close to the optimal initial cap found before, i.e. $C = NAV(t)/(1+1/\lambda)$. For the CRRA utility function, the cushion is also likely to be capped, with its maximum likely to decrease with the time to target date as shown in figure 1. The allocation to the risky asset is thus likely to decrease when getting closer to the target date following the *glide-path* principle of lifecycle strategies.

The program is investigated more rigorously in the next section. The results discussed so far are indeed based on a *myopic* view as they assume no change in the protection. And changing the protection in the future *does* affect the management of the cushion.

Optimal dynamic strategy allowing for increases in the protection

The management of the cushion and the estimation of the optimal protection have so far been considered independently and in a *myopic* sense. The more general problem, wherein the protection can be increased at any time, requires solving a partial differential equation (PDE) for the utility at the target date. Let us consider the case of the CRRA utility function. The utility at the target date depends on two state variables, $G(T)$ and $C(T)$. Let us re-write the utility in such way so as have only one variable to diffuse in the PDE:

$$U[G(T),C(T)] = \frac{[G(T)]^{\gamma}}{\gamma} + \lambda \frac{[NAV(T)-G(T)]^{\gamma}}{\gamma} = \frac{[G(T)]^{\gamma}}{\gamma} \cdot \left[1 + \lambda \left(\frac{NAV(T)}{G(T)} - 1 \right)^{\gamma} \right] \quad (22)$$

Let us now define $x = NAV/G$ and $f(x) = (x-1)^{\gamma}$. Increasing the protection from G_{old} to G_{new} decreases x to $x - \varepsilon$ so that:

$$\begin{cases} \dfrac{G_{new}}{G_{old}} = \dfrac{NAV/G_{old}}{NAV/G_{new}} = \dfrac{x}{x-\varepsilon} \\ U_{new} = U_{old} \cdot \left(\dfrac{x}{x-\varepsilon} \right)^{\gamma} \dfrac{1+\lambda \cdot f(x-\varepsilon)}{1+\lambda \cdot f(x)} \end{cases} \quad (23)$$

As soon as x is too large, meaning that the NAV is too large compared to the protection, $U_{new} > U_{old}$ and the investor should then consider increasing the level of protection. The level of x above which the protection has to be increased is the constant x^* which can be found from solving the following equation:

$$\left(\frac{x}{x-\varepsilon}\right)^{\gamma}\left(1+\lambda\cdot f(x-\varepsilon)\right)-\left(1+\lambda\cdot f(x)\right)=0 \qquad (24)$$

As a result, the protection is increased in the diffusion process as soon as *x* is above the cap *x**: *x* is in this case brought back to *x**. As a result, the size of the cushion and the investment in the risky asset are also capped in the diffusion process on each date to take into account the future increases in the level of protection and the dynamics of the cushion. The program is solved in practice by adapting the utility function in the following way:

$$U[t,NAV(T),G(t,T)]=P[G(t,T)]+\lambda(t)\cdot E\{OP[NAV(T)-G(t,T)]\} \qquad (25)$$

where *t* in G(*t*,*T*) represents the time t when the protection (guarantee) is given. Note that the relative weight between protection and over-performance $\lambda(t)$ is now a function of time. The changes in the *NAV* are constrained by the fact that the utility function of the investor changes in the future taking into account the last given level of protection.

We shall now investigate the impact of these changes on the optimal strategy. We set γ = 0.50, σ = 15 percent and the Sharpe ratio = 0.40. For the *myopic* case we used λ = 0.25. In the non-*myopic* case we considered both $\lambda(t)$ = 0.25, constant over time, and $\lambda(t)$ = 0.25 + 0.02 * (*T*- *t*), where the longer before the target date the more $\lambda(t)$ is larger than 0.25. We shall consider these examples both with the exposure to the risky asset capped at 100% of the *NAV* and with no maximum exposure.

In figure 3 we show the maximum level of the cushion as a function of the time to target date. The protection would be increased should the cushion increase above this maximum.

Let us start with the *myopic* case. The introduction of a maximum exposure strongly reduces the expected growth of the cushion so that the proportion of the capital allocated to the cushion decreases to the benefit of the protection level.

The effect of changing from the myopic case to the non-*myopic* case while keeping the same λ definition is even stronger, as the future cap on the cushion even prevents the strategy from reaching the maximum exposure. The maximum cushion is thus independent of whether a maximum exposure was imposed or not. This motivates the introduction of the time dependent $\lambda(t)$ in the non-*myopic* case. Knowing that the protection is likely to increase with time, the investor increases the weight of over-performance in the utility function when there is a longer time left to target date.

In figure 4 we show the optimal protection ratio, i.e. the minimum level of protection compared to the *NAV*, as a function of the time left to target date for the case of time dependent $\lambda(t) = 0.25 + 0.02 * (T\text{-}t)$ with the maximum exposure at 100%. Again we set $\gamma = 0.50$, $\sigma = 15$ percent and the Sharpe ratio = 0.40.

The optimal protection increases with the interest rate. The optimal protection is also larger longer before the target date (provided the interest rate is not too low). The optimal protection is well above 100% in many cases. Note finally that a very low interest rate leads the optimal protection ratio to be a decreasing function of the time to target date and optimal protection below 100%.

The bottom line is that target date funds should follow the dynamic strategy of cushion management and increasing protection shown in figures 3 and 4. We call this strategy Portfolio Insurance With Adaptive Protection, or PIWAP, because the minimum protection shown in figure 4 (and related to the cap of the cushion shown in figure 3) changes with interest rates and with the time left to target date.

CONCLUSIONS

The goal of this paper is to provide insight into the optimal parameterization of CPPI strategies based on the optimal trade-off between upside potential and capital protection at the target date. The main innovation is the use of a framework based on utility functions where the trade-off between upside potential and protection at maturity is explicitly modeled. Greater protection at the target date is important for investors as it increases the certainty of the outcome with a well-defined limit on eventual losses.

The approach is new since in previous literature the protection was seen as a minimum subsistence level: the optimal strategy is then a CPPI strategy with the protection at target date directly linked to the minimum subsistence level. In that sense, there is no clear rationale in that literature for increasing the protection at maturity given that the level of protection is more or less set exogenously by the subsistence level.

The second innovation of the paper is the optimal strategy proposed. The framework gives rise to a CPPI-like strategy but with a cap on the cushion, i.e. the level of protection at maturity is dynamically increased so that the cushion does not increase too much. The underlying intuition is straightforward: should the cushion increase too much, the upside potential would become very large – too large compared to the protection at target date. Higher utility would then be obtained by increasing the protection rather than letting the cushion drift higher.

Not new, but consistent with previous results in the literature, is the fact that the cushion is shown to be optimally invested by using a multiplier proportional to the ratio of the Sharpe ratio to the volatility of the risky asset. Here again the intuition is simple: should the volatility of the risky asset be too large, one should not invest too much in it given that CPPI strategies are pro-cyclical (sell low and buy high) and that too much volatility would therefore increase the cost of protection and reduce the upside potential. This effect is of course partially offset by a larger return per unit of risk (Sharpe ratio). The optimal leverage is close to the one used by practitioners e.g. the leverage is 3 for a Sharpe Ratio of 0.45 and a volatility of 15%.

This framework can be used to find theoretical answers to the frequent questions of sellers of portfolio insurance. The dilemma of expected upside potential at an attractive level of protection is well addressed and results are quite intuitive.

This new framework also enables the issue of interest rate hedging to be dealt with. If the protection is set at the time $t = 0$, hedging the bond floor against changes in the interest rate is optimal because of the concavity of the utility function on the cushion. If the protection can be increased after the time $t = 0$, this result no longer holds. Only partial hedging makes sense in order to avoid over-increasing the protection following changes in the interest rate. This issue is not dealt with here and requires further research as we considered only perfect hedging.

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